Mathematical and Logical Foundations of DBMS

November 6, 2023

Clearing Up Keys

The candidate key meets two conditions

- It is unique: Each key value uniquely identifies one record within the table, different tuples must not have identical keys
- It is minimal: if the key is a combination of attributes nothing from that combination can be removed without eliminating unique identification

Clearing Up Keys

- ALL candidate keys are superkeys (we are going to do some set theory today, candidate keys are a SUBSET of candidate keys)
- Any candidate key could be a primary key but we might choose to not use it

Here is an Example Where We have Both

StudentID	SocialSecurityNumber	FirstName	LastName
1	123-45-6789	John	Smith
2	987-65-4321	Alice	Johnson
3	123-45-6788	Bob	Brown
4	555-12-3456	Carol	Davis

Table 1: Example of a "Students" Table

Mathematical and Logical Foundations of DBMS

- We learned yesterday that database management - specifically the relational model - were able to revolutionize how databases were managed in the 1980s after innovations from IBM

Mathematical and Logical Foundations of DBMS

- We learned yesterday that database management - specifically the relational model - were able to revolutionize how databases were managed in the 1980s after innovations from IBM
- The programming languages they used were based on Codds Relational model

Mathematical and Logical Foundations of DBMS

- We learned yesterday that database management - specifically the relational model - were able to revolutionize how databases were managed in the 1980s after innovations from IBM
- The programming languages they used were based on Codds Relational model
- Of course, the relational model was born out of set theory

Mathematical and Logical Foundations of DBMS

- Why learn about set theory?

Mathematical and Logical Foundations of DBMS

- Why learn about set theory?
  - Would it make sense to learn to comprehend a spoken language without knowing grammar?

Mathematical and Logical Foundations of DBMS

- Why learn about set theory?
  - Would it make sense to learn to comprehend a spoken language without knowing grammar?
  - It is the method to the madness

Mathematical and Logical Foundations of DBMS

- Why learn about set theory?
  - Would it make sense to learn to comprehend a spoken language without knowing grammar?
  - It is the method to the madness
  - The logical arguments have direct implications how data is stored, queried, and joined

Mathematical and Logical Foundations of DBMS

- Why learn about set theory?
  - Would it make sense to learn to comprehend a spoken language without knowing grammar?
  - It is the method to the madness
  - The logical arguments have direct implications how data is stored, queried, and joined
  - Cartesian products, unions, differences, the inclusion exclusion principle, and more are all the basis for how data is joined in a way that is efficient and accurate

Mathematical and Logical Foundations of DBMS

- Why learn about set theory?

Mathematical and Logical Foundations of DBMS

- Why learn about set theory?
  - Yesterday, we learned that rows of datasets are called tuples
  - Each tuple is a collection of information, and may be considered a set

Mathematical and Logical Foundations of DBMS

- Why learn about set theory?
  - Yesterday, we learned that rows of datasets are called tuples
  - Each tuple is a collection of information, and may be considered a set
  - Each arbitrary cell in a database can be thought of as an element in a set

Mathematical and Logical Foundations of DBMS

- Our goal is to understand the logical and mathematical principals that underpin the rest of the class

Mathematical and Logical Foundations of DBMS

- Our goal is to understand the logical and mathematical principals that underpin the rest of the class
- If the logical rules of set theory are comprehensible and easy to understand, much of the rest of the course follows

Mathematical and Logical Foundations of DBMS

- Our goal is to understand the logical and mathematical principals that underpin the rest of the class
- If the logical rules of set theory are comprehensible and easy to understand, much of the rest of the course follows
- Feeling comfortable working with sets, and thinking about them in the abstract, is the same task as working with large datasets that you can't observe

Mathematical and Logical Foundations of DBMS

- Our goal is to understand the logical and mathematical principals that underpin the rest of the class
- If the logical rules of set theory are comprehensible and easy to understand, much of the rest of the course follows
- Feeling comfortable working with sets, and thinking about them in the abstract, is the same task as working with large datasets that you can't observe
- Efficiently reading the notation will also allow you to feel more comfortable reading academic texts to stay up to date on your own time during your work life to keep pace with new innovations or teach yourself on the fly when needed

Mathematical and Logical Foundations of DBMS

- Our goal is to understand the logical and mathematical principals that underpin the rest of the class
- If the logical rules of set theory are comprehensible and easy to understand, much of the rest of the course follows
- Feeling comfortable working with sets, and thinking about them in the abstract, is the same task as working with large datasets that you can't observe
- Efficiently reading the notation will also allow you to feel more comfortable reading academic texts to stay up to date on your own time during your work life to keep pace with new innovations or teach yourself on the fly when needed
- Will help you feel trained to be a chef, rather than a cook.

Mathematical and Logical Foundations of DBMS

- You will have the slides to work with, but taking notes will help
- You'll remember things better if you have something hand written
- Feel free to verbally interrupt of something doesn't make sense or if I am speaking too quickly

Logic Operators

- Conjunction $\wedge$ (1 and 2 $= 1 \wedge 2$)

Logic Operators

- Conjunction $\wedge$ (1 and 2 $= 1 \wedge 2$)
- Disjunction $\vee$ (1 or 2 $= 1 \vee 2$)

Logic Operators

- Conjunction $\wedge$ (1 and 2 $= 1 \wedge 2$)
- Disjunction $\vee$ (1 or 2 $= 1 \vee 2$)
- Negation $\neg$ (not 1 is $\neg$ 1)

Logic Operators

- Conjunction $\wedge$ (1 and $2 = 1 \wedge 2$)
- Disjunction $\vee$ (1 or $2 = 1 \vee 2$)
- Negation $\neg$ (not 1 is $\neg$ 1)
- Existential quantification ($\exists$)

Logic Operators

- Conjunction $\wedge$ (1 and 2 $= 1 \wedge 2$)
- Disjunction $\vee$ (1 or 2 $= 1 \vee 2$)
- Negation $\neg$ (not 1 is $\neg$ 1)
- Existential quantification ($\exists$)
- Logical equivalence $\Leftrightarrow$

Logic Operators

- Conjunction $\wedge$ (1 and 2 $= 1 \wedge 2$)
- Disjunction $\vee$ (1 or 2 $= 1 \vee 2$)
- Negation $\neg$ (not 1 is $\neg$ 1)
- Existential quantification ($\exists$)
- Logical equivalence $\Leftrightarrow$
- Such that :

Logic Operators

- Conjunction $\wedge$ (1 and 2 $= 1 \wedge 2$)
- Disjunction $\vee$ (1 or 2 $= 1 \vee 2$)
- Negation $\neg$ (not 1 is $\neg$ 1)
- Existential quantification ($\exists$)
- Logical equivalence $\Leftrightarrow$
- Such that :
- For all $\forall$

Sets and Elements

- Set theory is the study of the relationship between **sets**

Sets and Elements

- Set theory is the study of the relationship between **sets**
- A set is a structure, representing an unordered collection (group, plurality) of zero or more distinct (different) objects.

Sets and Elements

- Set theory is the study of the relationship between **sets**
- A set is a structure, representing an unordered collection (group, plurality) of zero or more distinct (different) objects.
- All sets are made from **elements**

Sets and Elements

- Set theory is the study of the relationship between **sets**
- A set is a structure, representing an unordered collection (group, plurality) of zero or more distinct (different) objects.
- All sets are made from **elements**
- Understanding how sets behave boils down to a focus on how their elements act

Some Famous Sets

- $\mathbb{N} = \{1, 2, 3, ...\}$ (set of natural numbers)

Some Famous Sets

- $\mathbb{N} = \{1, 2, 3, ...\}$ (set of natural numbers)
- $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$ (set of integers)

Some Famous Sets

- $\mathbb{N} = \{1, 2, 3, ...\}$ (set of natural numbers)
- $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$ (set of integers)
- $\mathbb{R} = \{x :$ x is a real number$\}$ (set of real numbers)

Sets and Elements

- We denote a set S by defining its elements

Sets and Elements

- We denote a set S by defining its elements
  - We can place these in $\{\}$

Sets and Elements

- We denote a set S by defining its elements
  - We can place these in $\{\}$
  - We may index elements of a set with subscripts

Sets and Elements

- We denote a set S by defining its elements
  - We can place these in $\{\}$
  - We may index elements of a set with subscripts
  - $\{s_1, s_2, s_3\}$

Sets and Elements

- Sets are unordered

Sets and Elements

- Sets are unordered
- Is $\{s_1, s_2, s_3\}$ the same as $\{s_3, s_2, s_1\}$?

Sets and Elements

- Sets are unordered
- Is $\{s_1, s_2, s_3\}$ the same as $\{s_3, s_2, s_1\}$?
- Yes!

Sets and Elements

- Capital letters name sets A, B, C. Conventionally, we would not name a set c or $larry$.

Sets and Elements

- Capital letters name sets A, B, C. Conventionally, we would not name a set c or $larry$.
- Commas separate elements of a set

Sets and Elements

- Capital letters name sets A, B, C. Conventionally, we would not name a set c or $larry$.
- Commas separate elements of a set
- $\{\}$ designate that the enclosed elements form a set

Sets and Elements

- Capital letters name sets A, B, C. Conventionally, we would not name a set c or $larry$.
- Commas separate elements of a set
- $\{\}$ designate that the enclosed elements form a set
- A $= \{Cow, Sheep, \{Chicken, Turkey\}, Goat\}$

Sets and Elements

- Capital letters name sets A, B, C. Conventionally, we would not name a set c or $larry$.
- Commas separate elements of a set
- $\{\}$ designate that the enclosed elements form a set
- A $= \{Cow, Sheep, \{Chicken, Turkey\}, Goat\}$
- We call this the roster method

Sets and Elements

- We use $\in$ to indicate that an element is an object of a set. It means something is an element of a set.

Sets and Elements

- We use $\in$ to indicate that an element is an object of a set. It means something is an element of a set.
- $\notin$ means the object is not an element of the set

Sets and Elements

- We use $\in$ to indicate that an element is an object of a set. It means something is an element of a set.
- $\notin$ means the object is not an element of the set
- True or false

Sets and Elements

- We use $\in$ to indicate that an element is an object of a set. It means something is an element of a set.
- $\notin$ means the object is not an element of the set
- True or false
- $3 \in \{1, 2, 3\}$

Sets and Elements

- We use $\in$ to indicate that an element is an object of a set. It means something is an element of a set.
- $\notin$ means the object is not an element of the set
- True or false
- $3 \in \{1, 2, 3\}$
- $\{z\} \in \{z, y, x, w\}$

Sets and Elements

- $W = \{x | x \text{is a day of the week}\}$

Sets and Elements

- $W = \{x | x \text{is a day of the week}\}$
- W is the set

Sets and Elements

- $W = \{x | x \text{is a day of the week}\}$
- W is the set
- x is the element

Sets and Elements

- $W = \{x | x \text{is a day of the week}\}$
- W is the set
- x is the element
- is a day of the week is the condition x must meet

Sets and Elements

- $W = \{x | x \text{is a day of the week}\}$
- W is the set
- x is the element
- is a day of the week is the condition x must meet
- $|$ means such that

Sets and Elements

- $W = \{x | x \text{is a day of the week}\}$
- W is the set
- x is the element
- is a day of the week is the condition x must meet
- $|$ means such that
- $\{\}$ denote set membership (elements in the set)

Sets and Elements

- $W = \{x | x \text{is a day of the week}\}$
- W is the set
- x is the element
- is a day of the week is the condition x must meet
- $|$ means such that
- $\{\}$ denote set membership (elements in the set)
- We call this set builder notation

Sets and Elements

- Two sets are equal **if and only if** they contain the exact same elements

Sets and Elements

- Two sets are equal **if and only if** they contain the exact same elements
- $\{x|x = 2\} = \{x \in \mathbb{N} \ |x < 3 \land x > 1\}$

Sets and Elements

- We can denote that an element is inside of a set with $\in$

Sets and Elements

- We can denote that an element is inside of a set with $\in$
    - $s_1 \in S$

Sets and Elements

- We can denote that an element is inside of a set with $\in$
    - $s_1 \in S$
    - Some sets are **empty**

Sets and Elements

- We can denote that an element is inside of a set with $\in$
    - $s_1 \in S$
    - Some sets are **empty**
    - $\emptyset$

Sets and Elements

- We can denote that an element is inside of a set with $\in$
    - $s_1 \in S$
    - Some sets are **empty**
    - $\emptyset$
    - $\{x|?\}$

Sets and Elements

- We can denote that an element is inside of a set with $\in$
  - $s_1 \in S$
  - Some sets are **empty**
  - $\emptyset$
  - $\{x|?\}$
  - A more formal statement is $\neg\exists x : x \in \emptyset$

Sets and Elements

- How big is a set? How many elements?
- We call that cardinality
- It is denoted as $||$
- Cardinality of the empty set $|\emptyset| = 0$
- Cardinality counts unique elements - nothing is counted twice
- $|\{1, 1, 2, 3\}| = 3$
- Today we deal with finite sets - cardinality being either 0 or a natural number

Sets and Elements

- U, or a Universal Set, is a set which has elements of all the related sets, without any repetition of elements

Sets and Elements

- Sets also have compliments

Sets and Elements

- Sets also have compliments
- Many ways to notate compliment of A

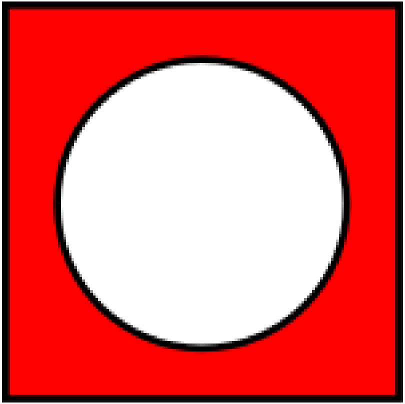
Sets and Elements

- Sets also have compliments
- Many ways to notate compliment of A
  - A^c
  - A'
  - $C(A)$
  - $\overline{A}$

Sets and Elements

- Sets also have compliments
- Many ways to notate compliment of A
  - A^c
  - A'
  - $C(A)$
  - $\overline{A}$
- $A^c = \{x \in U : x \notin A\}$

Sets and Elements



Sets and Elements

Two sets A and B are considered equal if and only if they have the same elements. In mathematical notation, we write this as:

$$A = B \iff (\forall x)(x \in A \iff x \in B)$$

Sets and Elements

Sets may also have subsets - smaller sets that “live” inside of them

Sets and Elements

Sets may also have subsets - smaller sets that “live” inside of them

- $A \subseteq B$

Sets and Elements

Sets may also have subsets - smaller sets that “live” inside of them

- $A \subseteq B$
- $A \supseteq B$ means $B \subseteq A$

Sets and Elements

Sets may also have subsets - smaller sets that "live" inside of them

- $A \subseteq B$
- $A \supseteq B$ means $B \subseteq A$
- Note $S = T \Leftrightarrow (S \subseteq T \wedge S \supseteq T)$

Sets and Elements

Sets may also have subsets - smaller sets that “live” inside of them

- $A \subseteq B$
- $A \supseteq B$ means $B \subseteq A$
- Note $S = T \Leftrightarrow (S \subseteq T \wedge S \supseteq T)$
- $\neg(S \subseteq T)$, means., $\exists x(x \in S \wedge x \notin T)$

Sets and Elements

$A \subset B$ (A is a proper subset of B) means that $A \subseteq B$ but $B \not\subseteq A$

Sets and Elements

$A \subset B$ (A is a proper subset of B) means that $A \subseteq B$ but $B \not\subseteq A$

- For example: $\{a_1, a_2\} \subset \{a_1, a_2, a_3\}$

Sets and Elements

- $\in$ is not the same as $\subseteq$
- $\in$ refers to elements, whereas $\subseteq$ refers to sets
- Recall the example about $\{4\}$

Sets and Elements

The objects that are elements of a set may themselves be sets. For example, let $S = \{x \mid x \subseteq \{1, 2, 3\}\}$, then

$S = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

Sets and Elements

- The power set $P(S)$ of a set S is the set of all subsets of S, denoted as $P(S) = \{x \mid x \subseteq S\}$.

Sets and Elements

- The power set $P(S)$ of a set S is the set of all subsets of S, denoted as $P(S) = \{x \mid x \subseteq S\}$.
- For example, if $S = \{a, b\}$, then $P(S) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.

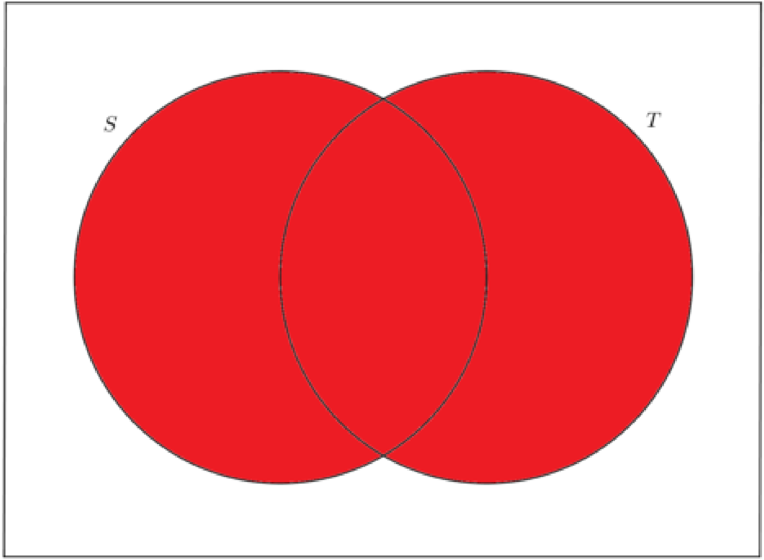
Remarkable Statements about Power sets

- Let S be a finite set with N elements. Then the powerset of P(S) (that is the set of all subsets of S) contains 2^N elements

Set Operations

- $\cup$
- $$X \cup Y = \{a : a \in X \vee a \in Y\}$$

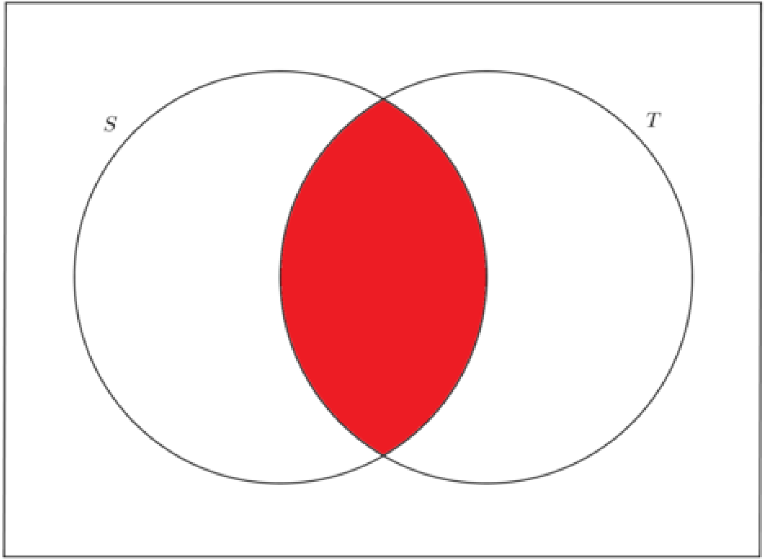
Set Operations



Set Operations

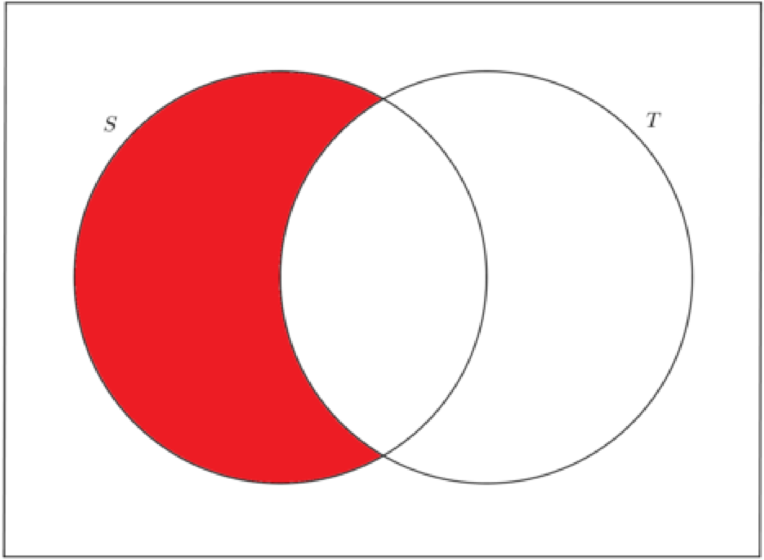
- $\cap$
- $A \cap B := x : x \in S \wedge x \in T$

Set Operations



Set Operations

- $\setminus$ OR $-$
- $S \setminus T = x : x \in S \land x \notin T$


S
T

Set Operations

For sets A and B, their Cartesian product $A \times B$ is defined as:

- $$A \times B := \{(a, b) \mid a \in A \wedge b \in B\}$$

Set Operations

For sets A and B, their Cartesian product $A \times B$ is defined as:

- $$A \times B := \{(a, b) \mid a \in A \wedge b \in B\}$$
- For example, if $A = \{a, b\}$ and $B = \{1, 2\}$, then $A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$.

How to Prove It

Proving Set Equality

- It is important to take an element focused perspective

Proving Set Equality

- It is important to take an element focused perspective
- Holistic perspective: $A \cup B$ is everything in A and everything in B

Proving Set Equality

- It is important to take an element focused perspective
- Holistic perspective: $A \cup B$ is everything in A and everything in B
- Elemental perspective: $x \in A \cup B$ iff $x \in A$ or $x \in B$.

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- The slide title is our claim that we will prove

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- The slide title is our claim that we will prove
- First step: translate it. Put it into words

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- The slide title is our claim that we will prove
- First step: translate it. Put it into words
- Who wants to try?

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- Next step: make it concrete! Abstractions are hard to work with at first

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- Next step: make it concrete! Abstractions are hard to work with at first
- How can we do that?

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- Next step: make it concrete! Abstractions are hard to work with at first
- How can we do that?
- Pictures
- Examples

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- $A = \{1, 2, 3\}$
- $B = \{1, 4, 5\}$
- $C = \{2, 3, 6\}$
- $D = \{1, 4, 5, 7\}$
- $E = \{2, 3, 6, 9\}$

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- $A = \{1, 2, 3\}$
- $B = \{1, 4, 5\}$
- $C = \{2, 3, 6\}$
- $D = \{1, 4, 5, 7\}$
- $E = \{2, 3, 6, 9\}$

- $D \cup E = \{\mathbf{1,2,3}, 4, 5, 6, 7, 9\}$

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- $A = \{1, 2, 3\}$
- $B = \{1, 4, 5\}$
- $C = \{2, 3, 6\}$
- $D = \{1, 4, 5, 7\}$
- $E = \{2, 3, 6, 9\}$

- $D \cup E = \{\mathbf{1{,}2{,}3}, 4, 5, 6, 7, 9\}$
- $A \subseteq D \cup E$

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- This should give us some confidence that what we are seeking to prove may be true

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- This should give us some confidence that what we are seeking to prove may be true
- Of course, this isn't a general proof because it is just one instance

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- This should give us some confidence that what we are seeking to prove may be true
- Of course, this isn't a general proof because it is just one instance
- We may be wrong!

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- This should give us some confidence that what we are seeking to prove may be true
- Of course, this isn't a general proof because it is just one instance
- We may be wrong!
- But we at least have some intuition now about where to start

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- Start writing: We will prove for any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- Start writing: We will prove for any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$
- We have made our goal clear

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- Start writing: We will prove for any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$
- We have made our goal clear
- We have also told the reader what to consider/our assumptions (these objects are sets, ect).

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- $A \subseteq B$ means that every element in A is also inside of B.

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- $A \subseteq B$ means that every element in A is also inside of B.
- We can prove $A \subseteq B$ by selecting an arbitrary $x \in A$ and then proving $x \in B$.

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- Let $x \in A$

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- Let $x \in A$
- We will prove $x \in D \cup E$

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- Let $x \in A$
- We will prove $x \in D \cup E$
- We introduced a new variable x

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- Let $x \in A$
- We will prove $x \in D \cup E$
- We introduced a new variable x
- It is arbitrary, so it is general, we didn't say imagine a prime number in A

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- We know that if A is a subset of B then every element in A is in B

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- We know that if A is a subset of B then every element in A is in B
- So by declaring that $x \in A$, we know that $x \in B \cup C$ by definition of subset

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- We know that if A is a subset of B then every element in A is in B
- So by declaring that $x \in A$, we know that $x \in B \cup C$ by definition of subset
- Notice here we are using a given fact rather than defining a new variable

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- So we know $x \in B \cup C$ by definition of subset

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- So we know $x \in B \cup C$ by definition of subset
- The Union of B and C is all the elements in both

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- So we know $x \in B \cup C$ by definition of subset
- The Union of B and C is all the elements in both
- So either $x \in B$ or $x \in C$

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- So we know $x \in B \cup C$ by definition of subset
- The Union of B and C is all the elements in both
- So either $x \in B$ or $x \in C$
- We cannot say for sure which is the case! So we consider both cases, and show our proof holds for either one

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- Case 1: $x \in B$

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- Case 1: $x \in B$
- $B \subseteq D$ given in the problem

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- Case 1: $x \in B$
- $B \subseteq D$ given in the problem
- Then $x \in D$ by definition of subset

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- Case 1: $x \in B$
- $B \subseteq D$ given in the problem
- Then $x \in D$ by definition of subset
- Case 2: $x \in C$

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- Case 1: $x \in B$
- $B \subseteq D$ given in the problem
- Then $x \in D$ by definition of subset
- Case 2: $x \in C$
- $C \subseteq E$

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- Case 1: $x \in B$
- $B \subseteq D$ given in the problem
- Then $x \in D$ by definition of subset
- Case 2: $x \in C$
- $C \subseteq E$
- Then $x \in E$ by definition of subset

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- Don't lose sight of the prize!

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- Don't lose sight of the prize!
- If $x \in D$, then $x \in D \cup E$
- If $x \in E$, then $x \in D \cup E$

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- Don't lose sight of the prize!
- If $x \in D$, then $x \in D \cup E$
- If $x \in E$, then $x \in D \cup E$
- In either case, $x \in D \cup E$, so $x \in A$ and $x \in D \cup E$

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- Don't lose sight of the prize!
- If $x \in D$, then $x \in D \cup E$
- If $x \in E$, then $x \in D \cup E$
- In either case, $x \in D \cup E$, so $x \in A$ and $x \in D \cup E$
- By definition of subset, $A \subseteq D \cup E$

Guided Example : For any sets A, B, C, D, E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$

- Don't lose sight of the prize!
- If $x \in D$, then $x \in D \cup E$
- If $x \in E$, then $x \in D \cup E$
- In either case, $x \in D \cup E$, so $x \in A$ and $x \in D \cup E$
- By definition of subset, $A \subseteq D \cup E$
- We are done!

Equality

- If you want to prove set equality (A=B) you need to establish $A \subseteq B$ and $B \subseteq A$.

Equality

- If you want to prove set equality (A=B) you need to establish $A \subseteq B$ and $B \subseteq A$.
- This makes intuitive sense: imagine these sets -

Equality

- If you want to prove set equality (A=B) you need to establish $A \subseteq B$ and $B \subseteq A$.
- This makes intuitive sense: imagine these sets -
- $A = \{Jeff, Stacy\}$B $= \{Jeff, Tom, Stacy\}$

Equality

- If you want to prove set equality (A=B) you need to establish $A \subseteq B$ and $B \subseteq A$.
- This makes intuitive sense: imagine these sets -
- $A = \{Jeff, Stacy\}$B $= \{Jeff, Tom, Stacy\}$
- If we stopped our proof at showing $A \subseteq B$ and claimed equality, we would be missing the fact that there is an element in B not in A, implying they are not equal

Equality

- If you want to prove set equality (A=B) you need to establish $A \subseteq B$ and $B \subseteq A$.
- This makes intuitive sense: imagine these sets -
- $A = \{Jeff, Stacy\}$B $= \{Jeff, Tom, Stacy\}$
- If we stopped our proof at showing $A \subseteq B$ and claimed equality, we would be missing the fact that there is an element in B not in A, implying they are not equal
- It is therefore important to show both sides of the equality are subsets of one another to do a complete equality proof

Guided Example: For sets A, B, $A \cup B = A$ iff $A \in P(B)$

- Let's follow our steps again: why, in words, would this logically hold?

Guided Example: For sets A, B, $A \cup B = A$ iff $A \in P(B)$

- Let's follow our steps again: why, in words, would this logically hold?
- Recall $P(S) = \{T | T \subset S\}$

Guided Example: For sets A, B, $A \cup B = A$ iff $A \in P(B)$

- Let's follow our steps again: why, in words, would this logically hold?
- Recall $P(S) = \{T | T \subset S\}$
- The Power set is a set made up of other sets

Guided Example: For sets A, B, $A \cup B = A$ iff $A \in P(B)$

- Let's follow our steps again: why, in words, would this logically hold?
- Recall $P(S) = \{T | T \subset S\}$
- The Power set is a set made up of other sets
- In words then, what are we saying?

Guided Example: For sets A, B, $A \cup B = A$ iff $A \in P(B)$

- Let's follow our steps again: why, in words, would this logically hold?
- Recall $P(S) = \{T | T \subset S\}$
- The Power set is a set made up of other sets
- In words then, what are we saying?
- If the elements that are in A and B are equal to A, then A is part of the Power Set of B, meaning A is one of the group of all subsets of B

Guided Example: For sets A, B, $A \cup B = A$ iff $A \in P(B)$

- $A = \{q, r\}$

Guided Example: For sets A, B, $A \cup B = A$ iff $A \in P(B)$

- $A = \{q, r\}$
- $B = \{q, r, s\}$

Guided Example: For sets A, B, $A \cup B = A$ iff $A \in P(B)$

- $A = \{q, r\}$
- $B = \{q, r, s\}$
- $P(B) = \{\{\}, \{q\}, \{r\}, \{s\}, \{q, r\}, \{q, s\}, \{r, s\}, \{q, r, s\}\}$

Guided Example: For sets A, B, $A \cup B = A$ iff $A \in P(B)$

- This is a biconditional statement (if and only if or iff)

Guided Example: For sets A, B, $A \cup B = A$ iff $A \in P(B)$

- This is a biconditional statement (if and only if or iff)
- This means we need to prove that when $A \cap B = A$ that $A \in P(B)$

Guided Example: For sets A, B, $A \cup B = A$ iff $A \in P(B)$

- This is a biconditional statement (if and only if or iff)
- This means we need to prove that when $A \cap B = A$ that $A \in P(B)$
- AND that when $A \in P(B)$, $A \cap B = A$

Guided Example: For sets A, B, $A \cup B = A$ iff $A \in P(B)$

- This is a biconditional statement (if and only if or iff)
- This means we need to prove that when $A \cap B = A$ that $A \in P(B)$
- AND that when $A \in P(B)$, $A \cap B = A$
- First statement implies the second, and vice versa

Guided Example: For sets A, B, $A \cup B = A$ iff $A \in P(B)$

- First, let's assume that $A \cap B = A$. We will prove $A \in P(B)$

Guided Example: For sets A, B, $A \cup B = A$ iff $A \in P(B)$

- First, let's assume that $A \cap B = A$. We will prove $A \in P(B)$
- Next, we assume $A \in P(B)$ We will prove $A \cap B = A$.

Guided Example: For sets A, B, $A \cup B = A$ iff $A \in P(B)$

- First, let's assume that $A \cap B = A$. We will prove $A \in P(B)$
- Next, we assume $A \in P(B)$ We will prove $A \cap B = A$.
- Notice our strategy: we assume part one, and use it to prove part two. Then we assume part two, and use it to prove part one.

Guided Example: For sets A, B, $A \cup B = A$ iff $A \in P(B)$

- First, let's assume that $A \cap B = A$. We will prove $A \in P(B)$
- Next, we assume $A \in P(B)$ We will prove $A \cap B = A$.
- Notice our strategy: we assume part one, and use it to prove part two. Then we assume part two, and use it to prove part one.
- For the first step we assume $A \cap B = A$. We don't need to prove it until step 2.

Guided Example: For sets A, B, $A \cup B = A$ iff $A \in P(B)$

- First, let's assume that $A \cap B = A$. We will prove $A \in P(B)$

Guided Example: For sets A, B, $A \cup B = A$ iff $A \in P(B)$

- First, let's assume that $A \cap B = A$. We will prove $A \in P(B)$
- Let $x \in A$, then $x \in A$ and $x \in B$ by definition of set intersection

Guided Example: For sets A, B, $A \cup B = A$ iff $A \in P(B)$

- First, let's assume that $A \cap B = A$. We will prove $A \in P(B)$
- Let $x \in A$, then $x \in A$ and $x \in B$ by definition of set intersection
- This means $x \in B$, so $x \in P(B)$ by definition of Power Set

Guided Example: For sets A, B, $A \cup B = A$ iff $A \in P(B)$

- First, let's assume that $A \cap B = A$. We will prove $A \in P(B)$
- Let $x \in A$, then $x \in A$ and $x \in B$ by definition of set intersection
- This means $x \in B$, so $x \in P(B)$ by definition of Power Set
- This completes step 1.

Guided Example: For sets A, B, $A \cup B = A$ iff $A \in P(B)$

- Next, we assume $A \in P(B)$ We will prove $A \cap B = A$.

Guided Example: For sets A, B, $A \cup B = A$ iff $A \in P(B)$

- Next, we assume $A \in P(B)$ We will prove $A \cap B = A$.
- We need to show equality

Guided Example: For sets A, B, $A \cup B = A$ iff $A \in P(B)$

- Next, we assume $A \in P(B)$ We will prove $A \cap B = A$.
- We need to show equality
- $A \subseteq A \cap B$ and $A \cap B \subseteq A$

Guided Example: For sets A, B, $A \cup B = A$ iff $A \in P(B)$

- Next, we assume $A \in P(B)$ We will prove $A \cap B = A$.
- We need to show equality
- $A \subseteq A \cap B$ and $A \cap B \subseteq A$
- Let any $x \in A$. We will show $x \in A \cap B$.

Guided Example: For sets A, B, $A \cup B = A$ iff $A \in P(B)$

- Next, we assume $A \in P(B)$ We will prove $A \cap B = A$.
- We need to show equality
- $A \subseteq A \cap B$ and $A \cap B \subseteq A$
- Let any $x \in A$. We will show $x \in A \cap B$.
- We have assumed $A \in P(B)$. By definition of power set, $A \subseteq B$.

Guided Example: For sets A, B, $A \cup B = A$ iff $A \in P(B)$

- Next, we assume $A \in P(B)$ We will prove $A \cap B = A$.
- We need to show equality
- $A \subseteq A \cap B$ and $A \cap B \subseteq A$
- Let any $x \in A$. We will show $x \in A \cap B$.
- We have assumed $A \in P(B)$. By definition of power set, $A \subseteq B$.
- If $A \subseteq B$, then $x \in B$ by definition of subset.

Guided Example: For sets A, B, $A \cup B = A$ iff $A \in P(B)$

- Next, we assume $A \in P(B)$ We will prove $A \cap B = A$.
- We need to show equality
- $A \subseteq A \cap B$ and $A \cap B \subseteq A$
- Let any $x \in A$. We will show $x \in A \cap B$.
- We have assumed $A \in P(B)$. By definition of power set, $A \subseteq B$.
- If $A \subseteq B$, then $x \in B$ by definition of subset.
- As such, $x \in A$ and $x \in B$.

Guided Example: For sets A, B, $A \cup B = A$ iff $A \in P(B)$

- Next, we assume $A \in P(B)$ We will prove $A \cap B = A$.
- We need to show equality
- $A \subseteq A \cap B$ and $A \cap B \subseteq A$
- Let any $x \in A$. We will show $x \in A \cap B$.
- We have assumed $A \in P(B)$. By definition of power set, $A \subseteq B$.
- If $A \subseteq B$, then $x \in B$ by definition of subset.
- As such, $x \in A$ and $x \in B$.
- Done! This is an easy case, since we only have one set on one side of the equality

Inclusion-Exclusion principle

- For two finite sets S and T $|S \cup T| = |S| + |T| - |S \cap T|$

Inclusion-Exclusion principle

- For two finite sets S and T $|S \cup T| = |S| + |T| - |S \cap T|$
- Why is this true?

Inclusion-Exclusion principle

- For two finite sets S and T $|S \cup T| = |S| + |T| - |S \cap T|$
- Why is this true?
- Lets say $|S| = 2, |T| = 1$ and they are disjoint we have $|S \cup T| = 3$.

Inclusion-Exclusion principle

- For two finite sets S and T $|S \cup T| = |S| + |T| - |S \cap T|$
- Why is this true?
- Lets say $|S| = 2, |T| = 1$ and they are disjoint we have $|S \cup T| = 3$.
- Since they are disjoint, we can express this as

Inclusion-Exclusion principle

- For two finite sets S and T $|S \cup T| = |S| + |T| - |S \cap T|$
- Why is this true?
- Lets say $|S| = 2, |T| = 1$ and they are disjoint we have $|S \cup T| = 3$.
- Since they are disjoint, we can express this as
- 3 = 2+1-0

Inclusion-Exclusion principle

- What if they aren't disjoint?

Inclusion-Exclusion principle

- What if they aren't disjoint?
- $S = \{1, 2, 3\}$

Inclusion-Exclusion principle

- What if they aren't disjoint?
- $S = \{1, 2, 3\}$
- $T = \{1, 4\}$

Inclusion-Exclusion principle

- What if they aren't disjoint?
- $S = \{1, 2, 3\}$
- $T = \{1, 4\}$
- $S \cup T = \{1, 1, 2, 3, 4\}$

Inclusion-Exclusion principle

- What if they aren't disjoint?
- $S = \{1, 2, 3\}$
- $T = \{1, 4\}$
- $S \cup T = \{1, 1, 2, 3, 4\}$
- $S \cap T = \{1\}$

Inclusion-Exclusion principle

- What if they aren't disjoint?
- $S = \{1, 2, 3\}$
- $T = \{1, 4\}$
- $S \cup T = \{1, 1, 2, 3, 4\}$
- $S \cap T = \{1\}$
- $4 = 3 + 2$ - 1

Inclusion-Exclusion principle

- We can extend this logic to include a third set

Inclusion-Exclusion principle

- We can extend this logic to include a third set
- $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

Quick Question

There is a party!

- You notice 10 people have white shirts, 8 have red shirts
- 4 people have black shoes and white shirts
- 3 have black shoes and red shirts
- 21 people have red shirts or white shorts or black shoes
- How many have black shoes?

Solution

We will use set theory rules to translate the words into an algebraic expression. First, define the sets.

- White shirts: W
- Red shirts: R
- Black shoes: B

Solution

Next, define the relationships

- Assume people only wear one shirt to a party, so $(R \cap W) = \emptyset$. Then the set of red shirt guests (R) is a complement to the set of white shirt guests (R^c).
- Following this assumption, it implies that there are guests wearing other color shirts because $|R \cup R^c| < |R \cup R^c \cup B|$
- $|B \cup R \cup R^c| = 21$
- $|R \cap B| = 3$
- $|R^c \cap B| = 4$

Solution

Now, lets use the inclusion-exclusion principle to solve the equation

Solution

Now, lets use the inclusion-exclusion principle to solve the equation

- $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

Solution

Now, lets use the inclusion-exclusion principle to solve the equation

- $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
- $|B \cup R \cup R^c| = |R| + |R^c| + |B| - |R \cap R^c| - |R \cap B| - |B \cap R^c| + |R \cap R^c \cap B|$

Solution

Now, lets use the inclusion-exclusion principle to solve the equation

- $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
- $|B \cup R \cup R^c| = |R| + |R^c| + |B| - |R \cap R^c| - |R \cap B| - |B \cap R^c| + |R \cap R^c \cap B|$
- $21 = 8 + 10 + |B| - 0 - 3 - 4 + 0$

Solution

Now, lets use the inclusion-exclusion principle to solve the equation

- $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
- $|B \cup R \cup R^c| = |R| + |R^c| + |B| - |R \cap R^c| - |R \cap B| - |B \cap R^c| + |R \cap R^c \cap B|$
- $21 = 8 + 10 + |B| - 0 - 3 - 4 + 0$
- $10 = |B|$

Cartesian Product is Not Commutative

- $A = \{a, b\}$ and $B = \{1, 2\}$, then $A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$
- BUT $B \times A = \{(1, a), (1, b), (2, a), (2, b)\}$
- This is because for non-empty A and B, if A contains an element x, in $A \times B$ there will be an ordered pair leading with x, but this will not be the case in the reverse such ordered pair.

Cartesian Product is Not Commutative

- Why the requirement that they are non-empty?

Cartesian Product is Not Commutative

- Why the requirement that they are non-empty?
- Because $A \times \emptyset = \emptyset$

Cartesian Product is Not Commutative

- Why the requirement that they are non-empty?
- Because $A \times \emptyset = \emptyset$
- Let's prove this by **contradiction**

Proof: $A \times \emptyset = \emptyset$

- Let's assume the opposite: $A \times \emptyset \neq \emptyset$

Proof: $A \times \emptyset = \emptyset$

- Let's assume the opposite: $A \times \emptyset \neq \emptyset$
- Then, the ordered pair $(x, y) \in A \times \emptyset$

Proof: $A \times \emptyset = \emptyset$

- Let's assume the opposite: $A \times \emptyset \neq \emptyset$
- Then, the ordered pair $(x, y) \in A \times \emptyset$
- From the definition of Cartesian product, this would mean $x \in A$ and $y \in \emptyset$

Proof: $A \times \emptyset = \emptyset$

- Let's assume the opposite: $A \times \emptyset \neq \emptyset$
- Then, the ordered pair $(x, y) \in A \times \emptyset$
- From the definition of Cartesian product, this would mean $x \in A$ and $y \in \emptyset$
- However, by definition of empty set $y \notin \emptyset$

Proof: $A \times \emptyset = \emptyset$

- Let's assume the opposite: $A \times \emptyset \neq \emptyset$
- Then, the ordered pair $(x, y) \in A \times \emptyset$
- From the definition of Cartesian product, this would mean $x \in A$ and $y \in \emptyset$
- However, by definition of empty set $y \notin \emptyset$
- It is a contradiction to hold $A \times \emptyset \neq \emptyset$, therefore $A \times \emptyset = \emptyset$

Proof: $A \times B = B \times A$

If A, B are sets, $A \times B = B \times A$ if and only if $A = B$ or either A or B are $\emptyset$.

- For an if and only if proof, we need to prove the claim going in both directions

Proof: $A \times B = B \times A$

If A, B are sets, $A \times B = B \times A$ if and only if $A = B$ or either A or B are $\emptyset$.

- For an if and only if proof, we need to prove the claim going in both directions
- That means we need to show it is the case that if $A \times B = B \times A$, our conditions must hold.

Proof: $A \times B = B \times A$

If A, B are sets, $A \times B = B \times A$ if and only if $A = B$ or either A or B are $\emptyset$.

- For an if and only if proof, we need to prove the claim going in both directions
- That means we need to show it is the case that if $A \times B = B \times A$, our conditions must hold.
- It **also** means that when our conditions hold, it implies our statement

Proof: $A \times B = B \times A$

- First half

Proof: $A \times B = B \times A$

- First half
- If $A = B$ then $A \times B = A \times A$ and $B \times A = A \times A$. Then, $A \times B = B \times A$ because that statement is the same as $A \times A = A \times A$ which is true by definition of identity

Proof: $A \times B = B \times A$

- First half
- If $A = B$ then $A \times B = A \times A$ and $B \times A = A \times A$. Then, $A \times B = B \times A$ because that statement is the same as $A \times A = A \times A$ which is true by definition of identity
- If $A = \emptyset$, then $A \times B = \emptyset$. Same goes for B. If that is the case, then $A \times B = B \times A$

Proof: $A \times B = B \times A$

- Second half

Proof: $A \times B = B \times A$

- Second half
- Let A and B be non-empty sets and let it be the case that $A \times B = B \times A$

Proof: $A \times B = B \times A$

- Second half
- Let A and B be non-empty sets and let it be the case that $A \times B = B \times A$
- Now we show if this statement holds, then A=B

Proof: $A \times B = B \times A$

- Second half
- Let A and B be non-empty sets and let it be the case that $A \times B = B \times A$
- Now we show if this statement holds, then A=B
- Let $x \in A$. Then there is a $y \in B$ s.t. $(x, y) \in (A \times B)$.

Proof: $A \times B = B \times A$

- Second half
- Let A and B be non-empty sets and let it be the case that $A \times B = B \times A$
- Now we show if this statement holds, then A=B
- Let $x \in A$. Then there is a $y \in B$ s.t. $(x, y) \in (A \times B)$.
- Since $A \times B = B \times A$, then $(x, y) \in (B \times A)$

Proof: $A \times B = B \times A$

- Second half
- Let A and B be non-empty sets and let it be the case that $A \times B = B \times A$
- Now we show if this statement holds, then A=B
- Let $x \in A$. Then there is a $y \in B$ s.t. $(x, y) \in (A \times B)$.
- Since $A \times B = B \times A$, then $(x, y) \in (B \times A)$
- By definition of Cartesian product, $x \in B$.

Proof: $A \times B = B \times A$

- Second half
- Let A and B be non-empty sets and let it be the case that $A \times B = B \times A$
- Now we show if this statement holds, then A=B
- Let $x \in A$. Then there is a $y \in B$ s.t. $(x, y) \in (A \times B)$.
- Since $A \times B = B \times A$, then $(x, y) \in (B \times A)$
- By definition of Cartesian product, $x \in B$.
- Since $x \in A$ and $x \in B$, then $A \subseteq B$. Similarly, $B \subseteq A$

Proof: $A \times B = B \times A$

- Second half
- Let A and B be non-empty sets and let it be the case that $A \times B = B \times A$
- Now we show if this statement holds, then A=B
- Let $x \in A$. Then there is a $y \in B$ s.t. $(x, y) \in (A \times B)$.
- Since $A \times B = B \times A$, then $(x, y) \in (B \times A)$
- By definition of Cartesian product, $x \in B$.
- Since $x \in A$ and $x \in B$, then $A \subseteq B$. Similarly, $B \subseteq A$
- $B = A$

Distributive Properties of Cartesian Products

Let A, B and C be sets. $A \times (B \cup C) = (A \times B) \cup (A \times C)$

Distributive Properties of Cartesian Products

Let A, B and C be sets. $A \times (B \cup C) = (A \times B) \cup (A \times C)$

- Let $(x, y) \in A \times (B \cup C)$. Then $x \in A$ and $y \in B$ or $y \in C$.

Distributive Properties of Cartesian Products

Let A, B and C be sets. $A \times (B \cup C) = (A \times B) \cup (A \times C)$

- Let $(x, y) \in A \times (B \cup C)$. Then $x \in A$ and $y \in B$ or $y \in C$.
- Case 1: $y \in B$

Distributive Properties of Cartesian Products

Let A, B and C be sets. $A \times (B \cup C) = (A \times B) \cup (A \times C)$

- Let $(x, y) \in A \times (B \cup C)$. Then $x \in A$ and $y \in B$ or $y \in C$.
- Case 1: $y \in B$
- Then $(x, y) \in (A \times B)$ so $(x, y) \in (A \times B) \cup (A \times C)$

Distributive Properties of Cartesian Products

Let A, B and C be sets. $A \times (B \cup C) = (A \times B) \cup (A \times C)$

- Let $(x, y) \in A \times (B \cup C)$. Then $x \in A$ and $y \in B$ or $y \in C$.
- Case 1: $y \in B$
- Then $(x, y) \in (A \times B)$ so $(x, y) \in (A \times B) \cup (A \times C)$
- Case 2: $(x, y) \in (A \times C)$

Distributive Properties of Cartesian Products

Let A, B and C be sets. $A \times (B \cup C) = (A \times B) \cup (A \times C)$

- Let $(x, y) \in A \times (B \cup C)$. Then $x \in A$ and $y \in B$ or $y \in C$.
- Case 1: $y \in B$
- Then $(x, y) \in (A \times B)$ so $(x, y) \in (A \times B) \cup (A \times C)$
- Case 2: $(x, y) \in (A \times C)$
- Then $(x, y) \in (A \times C)$ so $(x, y) \in (A \times B) \cup (A \times C)$

Distributive Properties of Cartesian Products

Let A, B and C be sets. $A \times (B \cup C) = (A \times B) \cup (A \times C)$

- Let $(x, y) \in A \times (B \cup C)$. Then $x \in A$ and $y \in B$ or $y \in C$.
- Case 1: $y \in B$
- Then $(x, y) \in (A \times B)$ so $(x, y) \in (A \times B) \cup (A \times C)$
- Case 2: $(x, y) \in (A \times C)$
- Then $(x, y) \in (A \times C)$ so $(x, y) \in (A \times B) \cup (A \times C)$
- Therefore $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$

Distributive Properties of Cartesian Products

Let A, B and C be sets. $A \times (B \cup C) = (A \times B) \cup (A \times C)$

Distributive Properties of Cartesian Products

Let A, B and C be sets. $A \times (B \cup C) = (A \times B) \cup (A \times C)$

- Let $(x, y) \in A \times (B \cup C)$. Then $x \in A$ and $y \in B$ or $y \in C$.

Distributive Properties of Cartesian Products

Let A, B and C be sets. $A \times (B \cup C) = (A \times B) \cup (A \times C)$

- Let $(x, y) \in A \times (B \cup C)$. Then $x \in A$ and $y \in B$ or $y \in C$.
- Case 1: $y \in B$

Distributive Properties of Cartesian Products

Let A, B and C be sets. $A \times (B \cup C) = (A \times B) \cup (A \times C)$

- Let $(x, y) \in A \times (B \cup C)$. Then $x \in A$ and $y \in B$ or $y \in C$.
- Case 1: $y \in B$
- Then $(x, y) \in (A \times B)$ so $(x, y) \in (A \times B) \cup (A \times C)$

Distributive Properties of Cartesian Products

Let A, B and C be sets. $A \times (B \cup C) = (A \times B) \cup (A \times C)$

- Let $(x, y) \in A \times (B \cup C)$. Then $x \in A$ and $y \in B$ or $y \in C$.
- Case 1: $y \in B$
- Then $(x, y) \in (A \times B)$ so $(x, y) \in (A \times B) \cup (A \times C)$
- Case 2: $(x, y) \in (A \times C)$

Distributive Properties of Cartesian Products

Let A, B and C be sets. $A \times (B \cup C) = (A \times B) \cup (A \times C)$

- Let $(x, y) \in A \times (B \cup C)$. Then $x \in A$ and $y \in B$ or $y \in C$.
- Case 1: $y \in B$
- Then $(x, y) \in (A \times B)$ so $(x, y) \in (A \times B) \cup (A \times C)$
- Case 2: $(x, y) \in (A \times C)$
- Then $(x, y) \in (A \times C)$ so $(x, y) \in (A \times B) \cup (A \times C)$

Distributive Properties of Cartesian Products

Let A, B and C be sets. $A \times (B \cup C) = (A \times B) \cup (A \times C)$

- Let $(x, y) \in A \times (B \cup C)$. Then $x \in A$ and $y \in B$ or $y \in C$.
- Case 1: $y \in B$
- Then $(x, y) \in (A \times B)$ so $(x, y) \in (A \times B) \cup (A \times C)$
- Case 2: $(x, y) \in (A \times C)$
- Then $(x, y) \in (A \times C)$ so $(x, y) \in (A \times B) \cup (A \times C)$
- Therefore $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$

Distributive Properties of Cartesian Products

Let A, B and C be sets. $A \times (B \cup C) = (A \times B) \cup (A \times C)$

- Let $(x, y) \in A \times (B \cup C)$. Then $x \in A$ and $y \in B$ or $y \in C$.
- Case 1: $y \in B$
- Then $(x, y) \in (A \times B)$ so $(x, y) \in (A \times B) \cup (A \times C)$
- Case 2: $(x, y) \in (A \times C)$
- Then $(x, y) \in (A \times C)$ so $(x, y) \in (A \times B) \cup (A \times C)$
- Therefore $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$
- Recall a complete proof must also show $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$ to establish equality

Practice Exercises

True or false (and provide a proof) Let D, E be two sets

$(D \setminus E) \cup E = D$

True or false (and provide a poof) Let D, E be two sets

$D \cap (D \cup E) = D$

Summary

- Set theory is the formal study of the relationship between collections of objects
- Database management is an application of this theory
- Understanding the abstract rules from set theory will provide us with a guide post to move forward
- The more comfortable you feel with the logical rules from set theory, the easier it will be to think about relationships, entities, and manipulating data to form queries
- We will now turn to practicing these questions in a more guided way